$\delta(\text{fix } f) = \text{fix } (\delta f \ (\text{fix } f))$

or, Static differentiation of monotone fixed points

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Slogan:
The derivative of a fixed point
is the fixed point of its derivative.

1 Preliminaries

I use capital letters A, B, P, Q to stand for posets. I write $P \to Q$ for the poset of *monotone* maps from P to Q. I write $\Box P$ for the discrete poset on P, where $x \leqslant y \iff x = y$.

Fix a poset P with a least element $\bot : P$ admitting a least-fixed-point operator $\text{fix} : (P \to P) \to P$ such that $\text{fix } f = f^i \bot$ for some $i : \mathbb{N}$.

Let us consider a higher-order language $\mathcal{L}$ which can, among perhaps many other things, express monotone maps on P. Let's assume we have a theory of changes for $\mathcal{L}$, à la "A Theory of Changes for Higher Order Languages" by Cai et al, PLDI 2014. We wish to extend $\mathcal{L}$ with fix, and find a theory of changes for the resulting $\mathcal{L}_{\text{fix}}$. In particular, how do we compute derivatives of fixed points? The answer is in the title:

$$\delta(\text{fix } f) = \text{fix } (\delta f \ (\text{fix } f))$$

The correctness criterion for δ on an expression e with (without loss of generality) one free variable x is:

$$e \{x \mapsto x \oplus dx\} = e \oplus \delta e \tag{1}$$

This article gives a proof that this holds for the provided definition of $\delta(\text{fix } f)$, namely that:

$$(\text{fix } f) \{x \mapsto x \oplus dx\} = \text{fix } f \oplus \text{fix } (\delta f \ (\text{fix } f))$$

2 Assumptions

I will assume $\Delta P = P$ and make use of this implicitly. This is a fairly strong assumption, but is true for seminaive Datalog (where P is roughly "finite sets of facts" or "relations") and in Datafun. I do not provide definitions of $\oplus, \ominus$ etc for P; I assume they are already a part of $\mathcal{L}$'s theory of changes. I will also assume the following lemmas:

Lemma 1 (Bottom changes)**.** $\bot : \Delta P$ is a valid change to any $x : P$, and $x \oplus \bot = x$.

NB. Not all values $dx : \Delta A$ are necessarily *valid* changes to a given $x : A$. For example, -3 is an invalid change to the natural number 0, because $0 + -3$ is not a natural number. (The Cai et al paper formalizes this point using a dependently-typed theory of changes. I will argue a bit more informally.)

Lemma 2 (Function changes)**.** If df is a valid change to $f : A \to B$ and dx is a valid change to $x : A$, then:

1. $df\ x\ dx$ is a valid change to $f\ x$.
2. $(f \oplus df)\ (x \oplus dx) = f\ x \oplus df\ x\ dx$

Lemma 3 (Curiously specific)**.** For $f : P \to P$, we have $f \leqslant f \oplus \delta f$.

The substance of this "curiously specific" lemma is that all derivatives of functions which we take fixed points of must be *increasing* changes. In Datafun this is true universally: all changes are increases.

3 Proof of correctness

Consider some expression $f : P \to P$ in one free variable x, whose derivative is $\delta f : \Box P \to \Delta P \to \Delta P$. We assume δf is a correct derivative, and therefore by equation 1:

$$(\text{fix}\ f)\ \{x \mapsto x \oplus dx\} = \text{fix}\ (f\ \{x \mapsto x \oplus dx\}) = \text{fix}\ (f \oplus \delta f)$$

Thus it suffices to show that:

$$\text{fix}\ (f \oplus \delta f) = \text{fix}\ f \oplus \text{fix}\ (\delta f\ (\text{fix}\ f))$$

To do this, we will need the corollary of one crucial lemma:

Lemma 4 (Iterated changes)**.** If df is a valid change to f and dx is a valid change to $x : P$, then for any $i : \mathbb{N}$,

$$(f \oplus df)^i\ (x \oplus dx) = f^i\ x \oplus (df\ (f^{i-1}\ x) \circ df\ (f^{i-2}\ x) \circ \ldots \circ df\ (f\ x) \circ df\ x)\ dx$$

Proof. Induct on i, applying Lemma 2. The base case is $x \oplus dx = x \oplus dx$. The inductive case is:

$$\begin{array}{lll} & (f \oplus df)^{i+1}\ (x \oplus dx) & \\ = & (f \oplus df)\ (f^i\ x \oplus (df\ (f^{i-1}\ x) \circ \ldots \circ df\ x)\ dx) & \text{by inductive hypothesis} \\ = & f^{i+1}\ x \oplus df\ (f^i\ x)\ ((df\ (f^{i-1}\ x) \circ \ldots \circ df\ x)\ dx) & \text{by Lemma 2} \\ = & f^{i+1}\ x \oplus (df\ (f^i\ x) \circ df\ (f^{i-1}\ x) \circ \ldots \circ df\ x)\ dx & \end{array}$$

□

Theorem 5. For any $i : \mathbb{N}$ and any valid change $e : \Delta P$ to $\text{fix } f$,

$$(f \oplus \delta f)^i \ (\text{fix } f \oplus e) = \text{fix } f \oplus (\delta f \ (\text{fix } f))^i \ e$$

Proof. Apply Lemma 4 with $x = \text{fix } f$ and $dx = e$, noting that $f^j \ (\text{fix } f) = \text{fix } f$ for any j. □

We'll also need a simple but cute lemma about monotone fixed points in general:

Lemma 6 (Jason Reed's lemma)**.** For any $f : P \to P$, if $a \leqslant \text{fix } f$ and $\text{fix } f = f^i \perp$ then $f^i \ a = \text{fix } f$.

Proof. Since $\perp \leqslant a$, by monotonicity $\text{fix } f = f^i \perp \leqslant f^i \ a$. Since $a \leqslant \text{fix } f$, by monotonicity $f^i \ a \leqslant f^i \ (\text{fix } f) = \text{fix } f$. So by antisymmetry $f^i \ a = \text{fix } f$.[1] □

Armed with these, we can prove our main result:

Theorem 7 (Correctness of $\delta(\text{fix } f)$)**.**

$$\text{fix } (f \oplus \delta f) = \text{fix } f \oplus \text{fix } (\delta f \ (\text{fix } f))$$

Proof. Pick i such that:

$$\begin{aligned} \text{fix } (f \oplus \delta f) &= (f \oplus \delta f)^i \perp \\ \text{fix } (\delta f \ (\text{fix } f)) &= (\delta f \ (\text{fix } f))^i \perp \end{aligned}$$

We can pick such an i because $\text{fix } f = f^i \perp$ for some "iteration count" i; and since $f^i \perp$ is a fixed point, we can increase i without changing its value. So we take the larger of the "iteration counts" for $f \oplus \delta f$ and $\delta f \ (\text{fix } f)$.

Then:

$$\begin{aligned} & \text{fix } (f \oplus \delta f) & \\ = \ & (f \oplus \delta f)^i \ (\text{fix } f) & \text{by Reed's Lemma 6, since fix } f \leqslant \text{fix } (f \oplus \delta f) \ (*) \\ = \ & (f \oplus \delta f)^i \ (\text{fix } f \oplus \perp) & \text{by Lemma 1} \\ = \ & \text{fix } f \oplus (\delta f \ (\text{fix } f))^i \perp & \text{by Thm. 5, as } \delta f \text{ is valid \& } \perp \text{ is valid by Lemma 1} \\ = \ & \text{fix } f \oplus \text{fix } (\delta f \ (\text{fix } f)) & \end{aligned}$$

Which is what we wished to show.

At (*), $\text{fix } f \leqslant \text{fix } (f \oplus \delta f)$ follows from monotonicity of fix applied to $f \leqslant f \oplus \delta f$ (Lemma 3).

□

[1] This is a special-case of a more general lemma, which you can find on MathOverflow.